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## ► To cite this version:

Mostafa Adimy, Khalil Ezzinbi, Aziz Ouhinou. New Variation of Constants Formula for Some Partial Functional Differential Equations with Infinite Delay. 2006. hal-00258402

**HAL Id: hal-00258402**

**<https://hal.science/hal-00258402>**

Preprint submitted on 22 Feb 2008

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# New Variation of Constants Formula for Some Partial Functional Differential Equations with Infinite Delay <sup>1</sup>

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## Abstract

In this work, we give a new variation of constants formula for some partial functional differential equations with infinite delay. We assume that the linear part is not necessarily densely defined and satisfies the known Hille-Yosida condition. When the phase space is a uniform fading memory space, we establish a spectral decomposition of the phase space. This allows us to study the existence of almost periodic solutions when the equation has a bounded solution on the half line  $\mathbb{R}^+$ .

**Key words and phrases:** Hille-Yosida operator, integral solutions, variation of constants formula, uniform fading memory space, almost periodic solutions.

## 1 Introduction

The aim of this work is to establish a new variation of constants formula and the existence of almost periodic solution for the following partial functional differential equation with infinite delay

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t), & t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{cases} \quad (1)$$

where  $A : D(A) \rightarrow X$  is a nondensely defined linear operator on a complex Banach space  $(X, |\cdot|)$ ,  $\mathcal{B}$  is a normed linear space of functions mapping  $(-\infty, 0]$  into  $X$  and satisfying some fundamental Axioms which are introduced later,  $x_t$  is an element of  $\mathcal{B}$  defined by

$$x_t(\theta) = x(t + \theta) \text{ for } \theta \in (-\infty, 0],$$

$L$  is a bounded linear operator from  $\mathcal{B}$  into  $X$ , and  $f$  is a continuous  $X$ -valued function on  $\mathbb{R}^+$ . We assume that  $A$  is a Hille-Yosida operator, which means that  $A$  satisfies the following assumption:

(H<sub>1</sub>) there exists  $\omega \in \mathbb{R}$  such that  $]\omega, +\infty[ \subset \rho(A)$ , and

$$\sup_{\lambda > \omega} |(\lambda - \omega)^n (\lambda I - A)^{-n}| < +\infty,$$

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<sup>1</sup>This research is supported by TWAS Grant under contract: No: 03-030 RG/MATHS/AF/AC

To appear in Journal of Mathematical Analysis and Applications

where  $\rho(A)$  is the resolvent set of the operator  $A$ . We will use this variation of constants formula to establish the existence of almost periodic solutions for equation (1).

Variation of constants formulas for partial functional differential equations plays an important role to study such equations, for more details about this topics, we refer to [24] and [13]. Recently, in [4], the authors established a new variation of constants formula for neutral partial functional differential equations. This formula has been used to get some behavior results for the solutions. In this work we will use the same method used in [4], we establish a new formula for partial functional differential equations with infinite delay whose linear part is not necessarily densely defined. One subject of this work is to use this formula for the existence of almost periodic solutions. More precisely, we establish the equivalence between the existence of an almost periodic solution for equation (1) and the existence of a bounded solution on  $\mathbb{R}^+$ . Recall that in [14], and [16] the authors established a new variation of constants formula for equation (1), where  $A$  is densely defined and generates a strongly continuous semigroup on  $X$ . In what follows, we give an outline of this formula. Let  $x_t(\sigma, \varphi)$  be the mild solution of equation (1). The authors proved that  $x_t(\sigma, \varphi)$  is represented by this formula

$$x_t(\sigma, \varphi) = U(t - \sigma)\varphi + \lim_{n \rightarrow \infty} \int_{\sigma}^t U(t - s) \Gamma^n f(s) ds, \text{ for } t \geq \sigma, \quad (2)$$

where  $\Gamma^n f(s)$  is defined by

$$(\Gamma^n f(s))(\theta) = \begin{cases} (n\theta + 1) f(s), & -\frac{1}{n} \leq \theta \leq 0, \\ 0, & \text{if } \theta \leq -\frac{1}{n}, \end{cases}$$

and  $(U(t))_{t \geq 0}$  is the solution semigroup of equation (1) with  $f = 0$ . The authors used the formula (2) and they established the existence of an almost periodic solution of equation (1) if there is at least one bounded mild solution on  $\mathbb{R}^+$ .

This work presents an extension of the works [14] and [16]. We prove that the density of  $D(A)$  is not needed to get a new variation of constants formula. Where  $\mathcal{B}$  is a uniform fading memory space, we establish a spectral decomposition of  $\mathcal{B}$  which allows us to study the existence of almost solutions.

The problem of finding periodic and almost periodic solutions of differential equations has been studied by several authors we refer to [5], [10], [15], [16], [17], [20] and the references therein. We recall some elementary results about differential equations in finite dimensional space. If we consider the following ordinary differential equation

$$\frac{d}{dt}x(t) = Bx(t) + g(t), \quad t \in \mathbb{R}, \quad (3)$$

where  $B$  is a  $n \times n$  matrix, and  $g$  is a continuous function from  $\mathbb{R}$  to  $X$ . Bohr and Neugebauer proved if  $f$  is almost periodic, then the existence of a bounded solution on  $\mathbb{R}^+$  of equation (3) is equivalent to the existence of an almost periodic solution of equation (3), for more details, we refer to [10]. Bohr and Neugebauer still holds for functional differential equations in finite dimensional

space and with finite delay: the existence of a bounded solution is equivalent to the existence of periodic or almost periodic solutions. Since for such equations, the solution becomes compact when  $t > r$ , for more details, we refer to [11]. The compactness is not true when the delay is infinite. In [23], the authors showed that the compactness remains true for partial functional differential equations with infinite delay. For partial functional differential equations, the phase space  $\mathcal{B}$  should be uniform fading memory space in order to get "best" information about solutions.

This work is organized as follows: in section 2, we give the fundamental definition of the phase space  $\mathcal{B}$  and we recall some results about the existence of solutions of Equation (1). In section 3, we establish a new variation of constants formula of Equation (1). In section 4, we study the spectral decomposition of the phase space. In section 5, we study the existence of almost periodic solutions. The last section is devoted to study the existence of almost periodic solutions for Lotka-Volterra model with diffusion.

## 2 Preliminary results

In this work, we employ an axiomatic definition of the phase space  $\mathcal{B}$  which has been introduced at first by Hale and Kato [12]. In the following, we assume that  $(\mathcal{B}, \|\cdot\|)$  is a normed space of functions mapping  $(-\infty, 0]$  into  $X$  satisfying the following fundamental axioms:

(A) : There exist a positive constant  $N$ , a locally bounded function  $M(\cdot)$  on  $[0, +\infty)$  and a continuous function  $K(\cdot)$  on  $[0, +\infty)$ , such that if  $x : (-\infty, a] \rightarrow X$  is continuous on  $[\sigma, a]$  with  $x_\sigma \in \mathcal{B}$ , for some  $\sigma < a$ , then for all  $t \in [\sigma, a]$ ,

- i)  $x_t \in \mathcal{B}$ ;
- ii)  $t \mapsto x_t$  is continuous with respect to  $\|\cdot\|$  on  $[\sigma, a]$ ;
- iii)  $N |x(t)| \leq \|x_t\| \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} |x(s)| + M(t - \sigma) \|x_\sigma\|$ .

(B) :  $\mathcal{B}$  is a Banach space.

We assume that

(D<sub>1</sub>) : if  $(\phi_n)_{n \geq 0}$  is a sequence in  $\mathcal{B}$  such that  $\phi_n \rightarrow 0$  in  $\mathcal{B}$  as  $n \rightarrow +\infty$ , then for all  $\theta \leq 0$ ,  $(\phi_n(\theta))_{n \geq 0}$  converges to 0 in  $X$ .

Let  $C((-\infty, 0], X)$  : be the space of continuous functions from  $(-\infty, 0]$  into  $X$ . We suppose the following assumptions:

(D<sub>2</sub>) :  $\mathcal{B} \subset C((-\infty, 0], X)$ ,

(D<sub>3</sub>) : there exists  $\lambda_0 \in \mathbb{R}$  such that, for all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda > \lambda_0$  and  $x \in X$ , we have that  $e^{\lambda \cdot} x \in \mathcal{B}$  and

$$K_0 := \sup_{\substack{\operatorname{Re} \lambda > \lambda_0, x \in X \\ x \neq 0}} \frac{\|e^{\lambda \cdot} x\|}{|x|} < \infty,$$

where  $(e^{\lambda \cdot} x)(\theta) = e^{\lambda \theta} x$ , for  $\theta \in (-\infty, 0]$  and  $x \in X$ .

In the whole of this work, we assume that  $A$  satisfies the assumption  $(\mathbf{H}_1)$ .

**Lemma 1.** [7, Example 14.5, Proposition 14.6, pp.319-320] Let  $A_0$  be the part of the operator  $A$  in  $D(A)$  which is given by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ A_0 x = Ax. \end{cases}$$

Then  $A_0$  generates a  $C_0$ - semigroup  $(T_0(t))_{t \geq 0}$  on  $\overline{D(A)}$ .

The following definition and results are taken from [2].

**Definition 2.** [2] A function  $u : \mathbb{R} \rightarrow X$  is called an integral solution of Equation (1) on  $\mathbb{R}^+$  if the following conditions hold

a)  $u$  is continuous on  $\mathbb{R}^+$ ,

b)  $u_0 = \phi$ ,

c)  $\int_0^t u(s) ds \in D(A)$ , for  $t \geq 0$ ,

d)  $u(t) = \phi(0) + A \int_0^t u(s) ds + \int_0^t Lu_s ds + \int_0^t f(s) ds$ , for  $t \geq 0$ .

If the operator  $A$  is densely defined, then the integral solution coincides with the mild solution given in [14].

**Theorem 3.** [2, pp. 336] Assume that  $\mathcal{B}$  satisfies  $(\mathbf{A})$  and  $(\mathbf{B})$ . Then for all  $\phi \in \mathcal{B}$  such that  $\phi(0) \in \overline{D(A)}$ , Equation (1) has a unique integral solution  $u(\cdot, \phi, L, f)$  on  $\mathbb{R}^+$  given by

$$u(t) = \begin{cases} T_0(t) \phi(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t T_0(t-s) \lambda R(\lambda, A) [Lu_s + f(s)] ds, & t \geq 0, \\ \phi(t), & t < 0. \end{cases}$$

A continuous function  $u$  on  $\mathbb{R}$  is said to be an integral solution of Equation (1) on  $\mathbb{R}$ , if  $u_s \in \mathcal{B}$  for all  $s \in \mathbb{R}$  and

$$u(t) = T_0(t - \sigma) u(\sigma) + \lim_{\lambda \rightarrow +\infty} \int_{\sigma}^t T_0(t-s) \lambda R(\lambda, A) [Lu_s + f(s)] ds, \quad t \geq \sigma.$$

### 3 The variation of constants formula

Let  $\mathcal{B}_A := \{\phi \in \mathcal{B} : \phi(0) \in \overline{D(A)}\}$  be the phase space corresponding to equation (1). Define  $U(t)$  for  $t \geq 0$  by

$$U(t) \phi = u_t(\cdot, \phi, L), \quad \phi \in \mathcal{B}_A,$$

where  $u(\cdot, \phi, L)$  is the integral solution of equation (1) with  $f = 0$ .

**Proposition 4.** [2, Proposition 2]  $(U(t))_{t \geq 0}$  is a  $C_0$ -semigroup on  $\mathcal{B}_A$ . That's

a)  $U(0) = Id$ ,

b)  $U(t+s) = U(t)U(s)$ , for  $t, s \geq 0$ ,

c) for all  $\phi \in \mathcal{B}_A$ ,  $t \mapsto U(t)\phi$  is continuous.

Moreover,  $(U(t))_{t \geq 0}$  satisfies the translation property

$$(U(t)\phi)(\theta) = \begin{cases} U(t+\theta)\phi(0), & t+\theta \geq 0 \\ \phi(t+\theta), & t+\theta < 0. \end{cases}$$

In order to establish a new variation of constants formula, we follow the same approach used in [4]. Before we need to recall the following results.

**Lemma 5.** [2, Proposition 5] Let  $\mathcal{B}$  satisfies Axioms **(A)**, **(B)**, **(D<sub>1</sub>)** and **(D<sub>2</sub>)**. Then the infinitesimal generator  $A_U$  of  $(U(t))_{t \geq 0}$  is given by:

$$\begin{cases} D(A_U) &= \left\{ \begin{array}{l} \phi \in C^1((-\infty, 0], X) \cap \mathcal{B}_A : \phi' \in \mathcal{B}_A, \phi(0) \in D(A) \text{ and} \\ \phi'(0) = A\phi(0) + L(\phi) \end{array} \right\}, \\ A_U\phi &= \phi'. \end{cases}$$

By Axiom **(D<sub>3</sub>)**, we define for each complex number  $\lambda$  such that  $\operatorname{Re}(\lambda) > \lambda_0$ , the linear operator  $\Delta(\lambda) : D(A) \rightarrow X$  by

$$\Delta(\lambda) = \lambda I - A - L(e^{\lambda \cdot} I).$$

Consider the space  $\mathfrak{X} := \mathcal{B}_A \oplus \langle X_0 \rangle$ , where  $\langle X_0 \rangle = \{X_0x : x \in X\}$  and  $X_0x$  is a function defined by

$$(X_0x)(\theta) = \begin{cases} 0 & \text{if } \theta \in (-\infty, 0), \\ x & \text{if } \theta = 0. \end{cases}$$

Then  $\mathfrak{X}$  endowed with the norm  $\|\phi + X_0x\| = \|\phi\| + |x|$  is a Banach space.

**Theorem 6.** Assume that  $\mathcal{B}$  satisfies Axioms **(A)**, **(B)**, **(D<sub>1</sub>)**, **(D<sub>2</sub>)** and **(D<sub>3</sub>)**. Then the extension  $\widetilde{A}_U$  of the operator  $A_U$  defined on  $\mathfrak{X}$  by

$$\begin{cases} D(\widetilde{A}_U) &= \{\phi \in \mathcal{B}_A : \phi' \in \mathcal{B}_A, \text{ and } \phi(0) \in D(A)\}, \\ \widetilde{A}_U\phi &= \phi' + X_0(A\phi(0) + L\phi - \phi'(0)), \end{cases}$$

is a Hille-Yosida operator on  $\mathfrak{X}$ .

For the proof we need the following fundamental lemma.

**Lemma 7.** There exist  $\omega_1 > \lambda_0$  and  $M_1 \in \mathbb{R}$  such that, for  $\lambda > \omega_1$ , we have

a)  $\Delta(\lambda)$  is invertible and  $|\Delta(\lambda)^{-1}| \leq \frac{M_0}{\lambda - \omega_1}$  ;

b)  $D(\widetilde{A}_U) = D(A_U) \oplus \langle e^{\lambda \cdot} \rangle$ , where

$$\langle e^{\lambda \cdot} \rangle = \{e^{\lambda \cdot}x : x \in D(A)\}.$$

c)  $\lambda \in \rho(\widetilde{A}_U)$ , and for  $n \in \mathbb{N}^*$ ,  $(\phi, x) \in \mathcal{B}_A \times X$ , one has

$$R(\lambda, \widetilde{A}_U)^n (\phi + X_0x) = R(\lambda, A_U)^n \phi + R(\lambda, A_U)^{n-1} (e^{\lambda \cdot} \Delta(\lambda)^{-1}x).$$

**Proof of the lemma.**

a) For  $\lambda > \bar{\omega} := \max\{0, \omega_0, \lambda_0\}$ , one has

$$\Delta(\lambda) = \lambda I - A - L(e^{\lambda \cdot} I) = (\lambda I - A) \left( I - R(\lambda, A) L(e^{\lambda \cdot} I) \right),$$

and

$$\left| R(\lambda, A) L(e^{\lambda \cdot} x) \right| \leq \frac{M_0 |L|}{\lambda - \omega_0} \|e^{\lambda \cdot} x\| \leq \frac{M_0 K_0 |L|}{\lambda - \omega_0} |x|, \quad x \in X.$$

Consequently

$$\left| R(\lambda, A) L(e^{\lambda \cdot} I) \right| \leq \frac{\bar{M}}{\lambda - \omega_0} < 1, \quad \text{for all } \lambda > \omega_1 := \bar{\omega} + \bar{M},$$

where  $\bar{M} := M_0 K_0 |L|$ . We conclude that the operator  $(I - R(\lambda, A) L(e^{\lambda \cdot} I))$  is invertible, and

$$\left| \left( I - R(\lambda, A) L(e^{\lambda \cdot} I) \right)^{-1} \right| \leq \frac{1}{1 - |R(\lambda, A) L(e^{\lambda \cdot} I)|} \leq \frac{\lambda - \omega_0}{\lambda - \omega_0 - \bar{M}}.$$

Consequently,  $\Delta(\lambda)$  is invertible for  $\lambda > \omega_1$  and

$$|\Delta(\lambda)^{-1}| \leq \frac{M_0}{\lambda - \bar{\omega}}.$$

b) Let  $\lambda > \omega_1$  and  $(e^{\lambda \cdot} x) \in D(A_U) \cap \langle e^{\lambda \cdot} \rangle$ . Then  $\lambda x = Ax + L(e^{\lambda \cdot} x)$ , with  $x \in D(A)$ . That is

$$\Delta(\lambda)x = 0.$$

Since  $\Delta(\lambda)$  is invertible for  $\lambda > \omega_1$ , we conclude that  $D(A_U) \cap \langle e^{\lambda \cdot} \rangle = \{0\}$ .

On the other hand, let  $\tilde{\psi} \in D(\widetilde{A_U})$  and  $\psi$  given by

$$\psi = \tilde{\psi} + e^{\lambda \cdot} \Delta(\lambda)^{-1} \left( A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0) \right)$$

Then,

$$\begin{aligned} A\psi(0) + L\psi &= A\tilde{\psi}(0) + L\tilde{\psi} + A\Delta(\lambda)^{-1} \left( A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0) \right) \\ &\quad + L \left( e^{\lambda \cdot} \Delta(\lambda)^{-1} \left( A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0) \right) \right), \\ &= A\tilde{\psi}(0) + L\tilde{\psi} - \Delta(\lambda)\Delta(\lambda)^{-1} \left( A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0) \right) \\ &\quad + \lambda\Delta(\lambda)^{-1} \left( A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0) \right), \\ &= \tilde{\psi}'(0) + \lambda\Delta(\lambda)^{-1} \left( A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0) \right), \\ &= \psi'(0). \end{aligned}$$

Hence  $\psi \in D(A_U)$ , which implies that  $D(\widetilde{A_U}) = D(A_U) \oplus \langle e^{\lambda \cdot} \rangle$ .

c) Let  $\lambda > \omega_1$  and  $\tilde{\psi} \in \mathfrak{X}$ . Then  $\tilde{\psi} = \psi + X_0 x$  for some  $\psi \in \mathcal{B}_A$  and  $x \in X$ . We seek for  $\tilde{\phi} = \phi + e^{\lambda \cdot} a \in D(\widetilde{A_U})$  such that  $(\lambda I - \widetilde{A_U}) \tilde{\phi} = \tilde{\psi}$ , where  $\phi \in D(A_U)$

and  $a \in D(A)$ . We have  $(\lambda I - \widetilde{A}_U)(\phi + e^{\lambda \cdot} a) = \psi + X_0 x$ , which is equivalent to find  $(a, \phi) \in D(A) \times D(A_U)$  such that

$$\begin{cases} (\lambda I - A_U) \phi &= \psi, \\ \Delta(\lambda) a &= x. \end{cases}$$

For  $\omega_1$  large enough, it follows that,  $(\lambda I - \widetilde{A}_U)^{-1}$  exists for  $\lambda > \omega_1$ , and

$$(\lambda I - \widetilde{A}_U)^{-1}(\psi + X_0 x) = (\lambda I - A_U)^{-1} \psi + e^{\lambda \cdot} \Delta(\lambda)^{-1} x.$$

Consequently, for  $n \in \mathbb{N}^*$ , we have

$$R(\lambda, \widetilde{A}_U)^n(\psi + X_0 x) = R(\lambda, A_U)^n \psi + R(\lambda, A_U)^{n-1} (e^{\lambda \cdot} \Delta(\lambda)^{-1} x). \quad \square$$

**Proof of Theorem 6.** Since  $A_U$  is the generator of the semigroup  $(U(t))_{t \geq 0}$  on  $\mathcal{B}_A$ , by Hille-Yosida's theorem [19] there exists a positive constant  $\tilde{M}$  such that

$$\sup_{n \in \mathbb{N}, \lambda > \omega_1} |(\lambda - \omega_1)^n R(\lambda, A_U)^n|_{L(\mathcal{B}_A)} \leq \tilde{M}.$$

By Lemma 7, there exist  $\omega_1$  and  $M_1 > 0$  such that

$$\sup_{n \in \mathbb{N}, \lambda > \omega_1} \left| (\lambda - \omega_1)^n R(\lambda, \widetilde{A}_U)^n \right| \leq M_1. \quad \square$$

**Lemma 8.** *The part of  $\widetilde{A}_U$  in  $\overline{D(\widetilde{A}_U)}$  is the operator  $A_U$ .*

**Proof.** From Lemma 5, the operator  $A_u$  generates a  $C_0$ -semigroup on  $\mathcal{B}_A$ , by Hille-Yosida's theorem  $\overline{D(A_U)} = \mathcal{B}_A$ . Since,  $D(A_U) \subset D(\widetilde{A}_U) \subset \mathcal{B}_A$ , then

$$\overline{D(A_U)} = \overline{D(\widetilde{A}_U)} = \mathcal{B}_A.$$

Let  $C$  be the part of  $\widetilde{A}_U$  in  $\overline{D(\widetilde{A}_U)}$ , which is defined by

$$\begin{cases} D(C) &= \left\{ \phi \in D(\widetilde{A}_U) : \widetilde{A}_U \phi \in \mathcal{B}_A \right\}, \\ C\phi &= \widetilde{A}_U \phi. \end{cases}$$

Then,  $D(A_U) \subseteq D(C)$  and  $A_U \phi = C\phi$  for all  $\phi \in D(A_U)$ .

Conversely, let  $\phi \in D(C)$ . Then

$$\begin{cases} \phi \in C^1((-\infty, 0], X) \cap \mathcal{B}_A, \phi' \in \mathcal{B}_A, \phi(0) \in D(A), \\ \phi' + X_0(A\phi(0) + L\phi - \phi'(0)) \in \mathcal{B}_A. \end{cases}$$

By assumption  $(D_2)$ , it follows that

$$\begin{cases} \phi \in D(\widetilde{A}_U) \text{ and } \phi'(0) = A\phi(0) + L\phi, \\ C\phi = \phi'. \end{cases}$$



From which we conclude that  $C = A_U$ .  $\square$

Consider the following evolution equation

$$\begin{cases} \frac{d}{dt}\xi(t) &= \widetilde{A}_U \xi(t) + X_0 f(t), \text{ for } t \geq 0 \\ \xi(0) &= \widetilde{\phi} \in \mathfrak{X}. \end{cases} \quad (4)$$

**Definition 9.** A continuous function  $\xi : [0, +\infty) \rightarrow \mathcal{B}_A$  is called an integral solution of Equation (4) if

- a)  $\int_0^t \xi(s) ds \in D(\widetilde{A}_U)$ , for  $t \geq 0$ ,
- b)  $\xi(t) = \widetilde{\phi} + \widetilde{A}_U \int_0^t \xi(s) ds + \int_0^t X_0 f(s) ds$ , for  $t \geq 0$ .

**Theorem 10.** Assume that  $(D_1)$ ,  $(D_2)$  and  $(D_3)$  hold. If  $u$  is an integral solution of Equation (1), then the function given by  $\xi(t) = u_t$ ,  $t \geq 0$ , is an integral solution of Equation (4) for  $\widetilde{\phi} = \phi$ . Conversely, if  $\xi$  is an integral solution of Equation (4) with  $\phi = \phi$ , then the function  $u$  defined by

$$u(t) = \begin{cases} \xi(t)(0), & \text{if } t \geq 0 \\ \phi(t), & \text{if } t \leq 0 \end{cases}$$

is an integral solution of Equation (1).

**Proof.** Let  $\phi \in \mathcal{B}_A$  and  $u$  be the integral solution of Equation (1). Define  $\xi : [0, \infty) \rightarrow \mathcal{B}_A$  by

$$\xi(t) = u_t, \text{ for } t \geq 0.$$

To compute the integral in  $\mathcal{B}$  in terms of the integral in  $X$ , we need the following lemma.

**Lemma 11.** [2] Assume that  $(D_1)$  holds, and  $F : [0, T] \rightarrow \mathcal{B}$  is continuous, then

$$\left( \int_0^T F(s) ds \right) (\theta) = \int_0^T F(s)(\theta) ds, \text{ for } \theta \leq 0.$$

By Lemma 11, we have

$$\begin{aligned} \frac{d}{d\theta} \left( \int_0^t u_s ds \right) (\theta) &= \frac{d}{d\theta} \left( \int_0^t u(s + \theta) ds \right) \\ &= \frac{d}{d\theta} \left( \int_\theta^{t+\theta} u(s) ds \right) \\ &= u_t(\theta) - \phi(\theta). \end{aligned}$$

Then

$$\widetilde{A}_U \left( \int_0^t \xi(s) ds \right) = u_t - \phi + X_0 \left( A \int_0^t u(s) ds + L \left( \int_0^t u_s ds \right) - u(t) - \phi(0) \right).$$

Since  $u$  is an integral solution of Equation (1), it follows that

$$u(t) = \phi(0) + A \int_0^t u(s) ds + L \left( \int_0^t u_s ds \right) + \int_0^t f(s) ds.$$

This implies that

$$\xi(t) = \phi + \widetilde{A}_U \int_0^t \xi(s) ds + X_0 \int_0^t f(s) ds, \text{ for } t \geq 0.$$

Consequently  $\xi$  is an integral solution of Equation (4). Conversely, let  $\xi$  be an integral solution of Equation (4) for  $\widetilde{\phi} = \phi$ . Then  $\xi$  satisfies the following translation property

$$\xi(t)(\theta) = \begin{cases} \xi(t+\theta)(0), & \text{if } t+\theta \geq 0, \\ \phi(t+\theta), & \text{if } t+\theta \leq 0. \end{cases}$$

In fact, for  $t+\theta \geq 0$ ,

$$\xi(t)(\theta) = (U(t)\phi)(\theta) + \lim_{\lambda \rightarrow +\infty} \int_0^t \left( U(t-s) \lambda R(\lambda, \widetilde{A}_U) X_0 f(s) \right) (\theta) ds.$$

Then

$$\begin{aligned} \xi(t)(\theta) &= (U(t+\theta)\phi)(0) + \lim_{\lambda \rightarrow +\infty} \int_0^{t+\theta} \left( U(t+\theta-s) \lambda R(\lambda, \widetilde{A}_U) X_0 f(s) \right) (0) ds + \\ &\quad \lim_{\lambda \rightarrow +\infty} \int_{t+\theta}^t \left( U(t-s) \lambda R(\lambda, \widetilde{A}_U) X_0 f(s) \right) (\theta) ds. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{\lambda \rightarrow +\infty} \int_{t+\theta}^t \left( U(t-s) \lambda R(\lambda, \widetilde{A}_U) X_0 f(s) \right) (\theta) ds \\ &= \lim_{\lambda \rightarrow +\infty} \int_{t+\theta}^t \left( \lambda R(\lambda, \widetilde{A}_U) X_0 f(s) \right) (t-s+\theta) ds \\ &= \lim_{\lambda \rightarrow +\infty} \int_{t+\theta}^t e^{\lambda(t-s+\theta)} \lambda \triangle (\lambda)^{-1} f(s) ds \\ &= 0. \end{aligned}$$

Which gives that

$$\begin{aligned} \xi(t)(\theta) &= (U(t+\theta)\phi)(0) + \lim_{\lambda \rightarrow +\infty} \int_0^{t+\theta} \left( U(t+\theta-s) \lambda R(\lambda, \widetilde{A}_U) X_0 f(s) \right) (0) ds \\ &= \xi(t+\theta)(0). \end{aligned}$$

If we consider the function

$$u(t) = \begin{cases} \xi(t)(0) & \text{if } t > 0, \\ \phi(t) & \text{if } t \leq 0. \end{cases}$$

Then  $\xi(t) = u_t$  for all  $t \geq 0$  and

$$u_t = \phi + \widetilde{A}_U \left( \int_0^t u_s ds \right) + \int_0^t X_0 f(s) ds, \text{ for } t \geq 0.$$

This implies that  $u$  is an integral solution of equation (1).  $\square$

**Theorem 12.** Assume that  $(D_1)$ ,  $(D_2)$ , and  $(D_3)$  hold. Then the integral solution  $x$  of Equation (1) is given by the following variation of constants formula

$$x_t = U(t)\phi + \lim_{n \rightarrow +\infty} \int_0^t U(t-s) \widetilde{B}_n(X_0 f(s)) ds, \quad t \geq 0, \quad (5)$$

where  $\widetilde{B}_n = n(nI - \widetilde{A}_u)^{-1}$ .

**Proof.**

This theorem is a consequence of Theorem 10 and the following lemma.

**Lemma 13.** [21] Let  $C$  be a Hille-Yosida operator on a Banach space  $Y$ , and  $\alpha : \mathbb{R}^+ \rightarrow Y$  be a continuous function. Consider the following problem

$$\begin{cases} \frac{d}{dt}x(t) &= Cx(t) + \alpha(t), \quad t \geq 0, \\ x(0) &= x_0 \in Y. \end{cases}$$

If  $x_0 \in \overline{D(C)}$ , then there exists a unique continuous function  $x$  such that

- a)  $\int_0^t x(s) ds \in D(C)$ ,  $t \geq 0$ ,
- b)  $x(t) = x_0 + C \int_0^t x(s) ds + \int_0^t \alpha(s) ds$ ,  $t \geq 0$ .

Moreover,  $x$  is given by

$$x(t) = S_0(t)x_0 + \lim_{\lambda \rightarrow +\infty} \int_0^t S_0(t-s) C_\lambda \alpha(s) ds, \quad t \geq 0,$$

where  $C_\lambda := \lambda(\lambda I - C)^{-1}$  and  $(S_0(t))_{t \geq 0}$  is the  $C_0$ -semigroup generated by the part of  $C$  in  $\overline{D(C)}$ .

## 4 Spectral decomposition of the phase space $\mathcal{B}_A$

Let  $C_{00}$  be the space of  $X$ -valued continuous function on  $] -\infty, 0]$  with compact support.

(C) : If a uniformly bounded sequence  $(\varphi_n)_{n \in \mathbb{N}}$  in  $C_{00}$  converges to a function  $\varphi$  compactly on  $(-\infty, 0]$ , then  $\varphi$  is in  $\mathcal{B}$  and  $\|\varphi_n - \varphi\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Let  $(S_0(t))_{t \geq 0}$  be the strongly continuous semigroup defined on the subspace

$$\mathcal{B}_0 := \{\phi \in \mathcal{B} : \phi(0) = 0\}$$

by

$$(S_0(t)\phi)(\theta) = \begin{cases} \phi(t+\theta) & \text{if } t+\theta \leq 0, \\ 0 & \text{if } t+\theta \geq 0. \end{cases}$$

**Definition 14.** Assume that the space  $\mathcal{B}$  satisfies Axioms (B) and (C),  $\mathcal{B}$  is said to be a fading memory space, if for all  $\phi \in \mathcal{B}_0$ ,

$$S_0(t)\phi \xrightarrow[t \rightarrow \infty]{} 0 \text{ in } \mathcal{B}_0.$$

Moreover,  $\mathcal{B}$  is said to be a uniform fading memory space, if

$$|S_0(t)| \xrightarrow[t \rightarrow \infty]{} 0.$$

**Lemma 15.** [13, pp 190] The following statements hold:

- a) If  $\mathcal{B}$  is a fading memory space, then the functions  $K(\cdot)$  and  $M(\cdot)$  in Axiom (A) can be chosen to be constants.
- b) If  $\mathcal{B}$  is a uniform fading memory space, then we can choose the function  $K(\cdot)$  constant and the function  $M(\cdot)$  such that  $M(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**Proposition 16.** [13] If the phase space  $\mathcal{B}$  is a fading memory space, then the space  $BC((-\infty, 0], X)$  of bounded continuous  $X$ -valued functions on  $(-\infty, 0]$  endowed with the uniform norm topology, is continuous embedding in  $\mathcal{B}$ . In particular  $\mathcal{B}$  satisfies (D<sub>3</sub>), for  $\lambda_0 > 0$ .

In this section, we assume that

(H<sub>2</sub>) :  $T_0(t)$  is compact on  $\overline{D(A)}$ , for each  $t > 0$ .

(H<sub>3</sub>) :  $\mathcal{B}$  is a uniform fading memory space.

Let  $V$  be a bounded subset of a Banach space  $Y$ , the Kuratowski measure of noncompactness  $\alpha(V)$  of  $V$  is given by

$$\alpha(v) = \inf \left\{ d > 0 \text{ such that there exists a finite number of sets } V_1, \dots, V_n \text{ with } \begin{array}{l} \text{diam}(V_i) \leq d \text{ such that } V \subseteq \bigcup_{i=1}^n V_i \end{array} \right\},$$

and for a bounded linear operator  $F$  on  $Y$ , we define  $|F|_\alpha$  by

$$|F|_\alpha = \inf \{k > 0 : \alpha(F(V)) \leq k\alpha(V), \text{ for all bounded set } V \text{ of } Y\}.$$

For a  $C_0$ -semigroup  $(S(t))_{t \geq 0}$ , we define the essential growth bound  $\omega_{ess}(S)$  by

$$\omega_{ess}(S) = \lim_{t \rightarrow \infty} \frac{1}{t} \log |S(t)|_\alpha.$$

**Theorem 17.** [5] Assume that  $\mathcal{B}$  satisfies Axioms (A), (B), (D<sub>1</sub>) and Assumptions (H<sub>2</sub>), (H<sub>3</sub>) hold. Then

$$\omega_{ess}(U) < 0.$$

From [9, Corollary IV.2.11], it follows that

$$\{\lambda \in \sigma(A_U) : \operatorname{Re}(\lambda) \geq 0\}$$

is a finite subset and  $\mathcal{B}_A$  is decomposed as follows:

$$\mathcal{B}_A = \mathcal{S} \oplus \mathcal{V},$$

where  $\mathcal{S}$ ,  $\mathcal{V}$  are two closed subspaces of  $\mathcal{B}_A$  which are invariant by  $(U(t))_{t \geq 0}$ . Let  $U^{\mathcal{S}}(t)$  be the restriction of  $U(t)$  on  $\mathcal{S}$ . Then there exist positive constants  $N$  and  $\mu$  such that

$$\|U^{\mathcal{S}}(t)\phi\| \leq Ne^{-\mu t} \|\phi\|, \text{ for all } \phi \in \mathcal{S}.$$

$\mathcal{V}$  is a finite dimensional space and the restriction  $U^{\mathcal{V}}(t)$  of  $U(t)$  on  $\mathcal{V}$  becomes a group. Let  $\Pi^{\mathcal{S}}$  and  $\Pi^{\mathcal{V}}$  denote the projections on  $\mathcal{S}$  and  $\mathcal{V}$  respectively and  $d = \dim \mathcal{V}$ . Take a basis  $\{\phi_1, \dots, \phi_d\}$  in  $\mathcal{V}$ , then there exists  $d$ -element  $\{\psi_1, \dots, \psi_d\}$  in the dual space  $\mathcal{B}_A^*$  of  $\mathcal{B}_A$ , such that  $\langle \psi_i, \phi_j \rangle = \delta_{ij}$ , where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and  $\psi_i = 0$  on  $\mathcal{S}$ , where  $\langle \cdot, \cdot \rangle$  denotes the canonical pairing between the dual space and the original space. Denote by  $\Phi := (\phi_1, \dots, \phi_d)$  and  $\Psi$  the transpose of  $(\psi_1, \dots, \psi_d)$ , in particular one has

$$\langle \Psi, \Phi \rangle = \mathbb{I}_{\mathbb{R}^d},$$

where  $\mathbb{I}_{\mathbb{R}^d}$  is the identity  $d \times d$  matrix. For each  $\phi \in \mathcal{B}_A$ ,  $\Pi^{\mathcal{V}}\phi$  is computed by:

$$\begin{aligned} \Pi^{\mathcal{V}}\phi &= \Phi \langle \Psi, \phi \rangle \\ &= \sum_{i=1}^d \langle \psi_i, \phi \rangle \phi_i. \end{aligned}$$

Let  $\zeta(t) := (\zeta_1(t), \dots, \zeta_d(t))$  be the component of  $\Pi^{\mathcal{V}}x_t$  in the basis vector  $\Phi$ . Then

$$\Pi^{\mathcal{V}}x_t = \Phi \zeta(t), \text{ and } \zeta(t) = \langle \Psi, x_t \rangle.$$

Since  $(U^{\mathcal{V}}(t))_{t \geq 0}$  is a group on a finite dimensional space  $\mathcal{V}$ , then there exists a  $d \times d$  matrix  $G$  such that

$$U^{\mathcal{V}}(t)\phi = \Phi e^{Gt} \langle \Psi, \phi \rangle, \text{ for all } t \in \mathbb{R} \text{ and } \phi \in \mathcal{V},$$

this means that

$$U^{\mathcal{V}}(t)\Phi = \Phi e^{Gt}, \text{ for all } t \in \mathbb{R}.$$

For  $n > \omega_1$  and  $i \in \{1, \dots, d\}$ , we define the functional  $x_n^{*i}$  by

$$\langle x_n^{*i}, x \rangle = \langle \psi_i, \widetilde{B}_n(X_0 x) \rangle, \text{ for all } x \in X.$$

Then,  $x_n^{*i}$  is a bounded linear operator on  $X$  with  $|x_n^{*i}| \leq K_0 M_1 |\psi_i|$ . Define the  $d$ -column vector  $x_n^*$  as an element of  $\mathcal{L}(X, \mathbb{R}^d)$  given by the transpose of  $(x_n^{*1}, \dots, x_n^{*d})$ . Then, for all  $n \geq 1$ ,  $x \in X$

$$\langle x_n^*, x \rangle = \langle \Psi, \widetilde{B}_n(X_0 x) \rangle \text{ and } \sup_{n \geq \omega_1} |x_n^*| \leq K_0 M_1 \sup_{i=1, \dots, d} |\psi_i| < \infty.$$

**Theorem 18.** *The sequence  $(x_n^*)_{n \geq 0}$  converges weakly in  $\mathcal{L}(X, \mathbb{R}^d)$ , in the sense that*

$$\langle x_n^*, x \rangle \xrightarrow{n \rightarrow \infty} \langle x^*, x \rangle, \text{ for all } x \in X.$$

**Proof.**

We need the following fundamental lemma.

**Lemma 19** (Banach-Alaoglu-Bourbaki). *Let  $Y$  be a separable Banach space and  $(\mathcal{K}_n)_{n \geq 0}$  be a bounded sequence in  $Y^*$ . Then there exist  $\mathcal{K} \in Y^*$  and a subsequence  $(\mathcal{K}_{n_k})_{k \geq 0}$  such that*

$$\langle \mathcal{K}_{n_k}, x \rangle \xrightarrow{k \rightarrow \infty} \langle \mathcal{K}, x \rangle, \text{ for all } x \in Y.$$

Let  $Y_0$  be any separable closed subspace of  $X$ . By Lemma 19, the restriction  $(x_n^{Y_0^*})_{n \geq 0}$  of  $(x_n^*)_{n \geq 0}$  in  $Y_0$  has a subsequence  $(x_{n_k}^{Y_0^*})_{k \geq 0}$  such that

$$\lim_{k \rightarrow \infty} \langle x_{n_k}^{Y_0^*}, y \rangle = \langle x^{Y_0^*}, y \rangle, \text{ for all } y \in Y_0,$$

where  $x^{Y_0^*} \in Y_0^*$ . We claim that the whole sequence  $(x_n^{Y_0^*})_{n \geq 0}$  converges weakly in  $Y_0^*$  to  $x^{Y_0^*}$ . We proceed by contradiction and assume that there exists a subsequence  $(x_{m_k}^{Y_0^*})_{k \geq 0}$  of  $(x_n^{Y_0^*})_{n \geq 0}$  such that  $x_{m_k}^{Y_0^*} \xrightarrow{k \rightarrow \infty} x_1^{Y_0^*}$  weakly in  $Y_0$ , with  $x^{Y_0^*} \neq x_1^{Y_0^*}$ . To conclude we need the following lemma.

**Lemma 20.** *For any continuous function  $h : \mathbb{R}^+ \rightarrow X$  one has:*

$$\lim_{n \rightarrow \infty} \int_0^t U^\nu(t-s) \Pi^\nu(\widetilde{B}_n(X_0 h(s))) ds = \Phi \lim_{n \rightarrow \infty} \int_0^t e^{(t-s)G} \langle x_n^*, h(s) \rangle ds.$$

**Proof of the lemma.**

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^t U^\nu(t-s) \Pi^\nu(\widetilde{B}_n(X_0 h(s))) ds \\ &= \lim_{n \rightarrow \infty} \int_0^t (U^\nu(t-s) \Phi) \langle \Psi, \widetilde{B}_n(X_0 h(s)) \rangle ds, \\ &= \lim_{n \rightarrow \infty} \int_0^t \Phi e^{(t-s)G} \langle x_n^*, h(s) \rangle ds, \\ &= \Phi \lim_{n \rightarrow \infty} \int_0^t e^{(t-s)G} \langle x_n^*, h(s) \rangle ds. \end{aligned} \quad \square$$

Let  $h(\cdot) = y$  for any  $y \in Y_0$ . Then

$$\int_0^t e^{(t-s)G} \langle x^{Y_0^*}, y \rangle ds = \int_0^t e^{(t-s)G} \langle x_1^{Y_0^*}, y \rangle ds, \text{ for any } y \in Y_0.$$

This is true if and only if  $\langle x^{Y_0^*}, y \rangle = \langle x_1^{Y_0^*}, y \rangle$ , for all  $y \in Y_0$ . Which gives a contradiction. Consequently the whole sequence  $(x_n^{Y_0^*})_{n \geq 0}$  converges weakly in  $\mathcal{L}(Y_0, \mathbb{R}^d)$  to  $x^{Y_0^*}$ .

Let  $Y_1$  be another separable closed space of  $X$ . Then the restriction  $(x_n^{Y_1^*})_{n \geq 0}$  of  $(x_n^*)_{n \geq 0}$  in  $Y_1$  converges weakly to some  $x^{Y_1^*} \in Y_1^*$ , and we get that  $x^{Y_0^*} = x^{Y_1^*}$  in  $Y_0 \cap Y_1$ . Since  $(x_n^*)_{n \geq 0}$  converges weakly in  $Y_0 \cap Y_1$ , and by the uniqueness of the limit we obtain that  $x^{Y_0^*} = x^{Y_1^*}$  in  $Y_0 \cap Y_1$ . Let  $x^*$  be the operator defined by

$$\langle x^*, x \rangle = \langle x^{Y^*}, x \rangle,$$

for any separable closed space  $Y$  of  $X$  such that  $x \in Y$ . Then  $x^*$  is well defined and belongs to  $\mathcal{L}(X, \mathbb{R}^d)$ . Moreover,

$$\langle x_n^*, x \rangle \xrightarrow{n \rightarrow \infty} \langle x^*, x \rangle, \text{ for all } x \in X. \quad \square$$

Consequently, we get the following.

**Corollary 21.** *For any continuous function  $h : [0, T] \rightarrow X$ , we have*

$$\lim_{n \rightarrow \infty} \int_0^t U^\mathcal{V}(t-s) \Pi^\mathcal{V} \left( \widetilde{B}_n(X_0 h(s)) \right) ds = \Phi \int_0^t e^{(t-s)G} \langle x^*, h(s) \rangle ds, \text{ for all } t \in [0, T].$$

**Theorem 22.** *Assume that (A), (B), (D<sub>1</sub>), (D<sub>2</sub>) and (H<sub>2</sub>) hold. Let  $u$  be an integral solution of Equation (1) on  $\mathbb{R}$ . Then  $\zeta(t) = \langle \Psi, u_t \rangle$ ,  $t \in \mathbb{R}$ , is a solution of the following ordinary differential equation*

$$\dot{\zeta}(t) = G\zeta(t) + \langle x^*, f(t) \rangle, \quad t \in \mathbb{R}. \quad (6)$$

*Conversely, if  $f$  is bounded and  $\zeta$  is a solution of (6), then the function*

$$\left( \Phi \zeta(t) + \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^\mathcal{S}(t-s) \Pi^\mathcal{S} \left( \widetilde{B}_n(X_0 f(s)) \right) ds \right) (0) \quad (7)$$

*is an integral solution of Equation (1) on  $\mathbb{R}$ .*

**Proof.**

Using the variation of constants formula (5), we obtain that for  $t \geq \sigma$

$$\begin{aligned} \langle \Psi, x_t \rangle &= \langle \Psi, U(t-\sigma)x_\sigma \rangle + \left\langle \Psi, \lim_{n \rightarrow +\infty} \int_\sigma^t U(t-s) \left( \widetilde{B}_n(X_0 f(s)) \right) ds \right\rangle, \\ &= e^{(t-\sigma)G} \langle \Psi, x_\sigma \rangle + \lim_{n \rightarrow \infty} \int_\sigma^t e^{(t-s)G} \left\langle \Psi, \left( \widetilde{B}_n(X_0 f(s)) \right) \right\rangle ds, \\ &= e^{(t-\sigma)G} \langle \Psi, x_\sigma \rangle + \lim_{n \rightarrow \infty} \int_\sigma^t e^{(t-s)G} \langle x_n^*, f(s) \rangle ds, \\ &= e^{(t-\sigma)G} \langle \Psi, x_\sigma \rangle + \int_\sigma^t e^{(t-s)G} \langle x^*, f(s) \rangle ds. \end{aligned}$$

This means that  $\zeta(t) = \langle \Psi, x_t \rangle$ ,  $t \in \mathbb{R}$  is a solution of the ordinary differential equation (6). Conversely, if we assume that  $f$  is bounded on  $\mathbb{R}$ , then Formula (7) is well defined, since the restriction of the solution semigroup on  $\mathcal{S}$  is exponentially stable. Let  $y$  be defined by

$$y(t) := \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t U^\mathcal{S}(t-s) \Pi^\mathcal{S} \left( \widetilde{B}_n(X_0 f(s)) \right) ds \text{ for } t \in \mathbb{R}.$$

Then for  $t \geq \sigma$ ,

$$\begin{aligned}
U^{\mathcal{S}}(t - \sigma)y(\sigma) &+ \lim_{n \rightarrow +\infty} \int_{\sigma}^t U^{\mathcal{S}}(t - s) \Pi^{\mathcal{S}} \left( \widetilde{B}_n(X_0 f(s)) \right) ds \\
&= \lim_{n \rightarrow +\infty} \left( \int_{-\infty}^{\sigma} U^{\mathcal{S}}(t - s) \Pi^{\mathcal{S}} \left( \widetilde{B}_n(X_0 f(s)) \right) ds + \right. \\
&\quad \left. \int_{\sigma}^t U^{\mathcal{S}}(t - s) \Pi^{\mathcal{S}} \left( \widetilde{B}_n(X_0 f(s)) \right) ds \right), \\
&= y(t).
\end{aligned} \tag{8}$$

Moreover, the solution  $\zeta$  of Equation (6) is given by

$$\zeta(t) = e^{(t-\sigma)G} \zeta(\sigma) + \int_{\sigma}^t e^{(t-s)G} \langle x^*, f(s) \rangle ds, \text{ for } t \geq \sigma.$$

Corollary 21, gives that

$$\Phi \zeta(t) = \Phi e^{(t-\sigma)G} \zeta(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t U^{\mathcal{V}}(t - s) \Pi^{\mathcal{V}} \left( \widetilde{B}_n(X_0 f(s)) \right) ds, \text{ for } t \geq \sigma,$$

and

$$\Phi \zeta(t) = U^{\mathcal{V}}(t - \sigma) \Phi \zeta(\sigma) + \lim_{n \rightarrow \infty} n \int_{\sigma}^t U^{\mathcal{V}}(t - s) \Pi^{\mathcal{V}} \left( \widetilde{B}_n(X_0 f(s)) \right) ds, \text{ for } t \geq \sigma. \tag{9}$$

Set  $\xi(t) = \Phi \zeta(t) + y(t)$  on  $\mathbb{R}$ , by (8) and (9). We obtain that

$$\begin{aligned}
\xi(t) &= U(t - \sigma) (\Phi \zeta(\sigma) + y(\sigma)) \\
&\quad + \lim_{n \rightarrow \infty} n \int_{\sigma}^t U(t - s) [\Pi^{\mathcal{V}} + \Pi^{\mathcal{S}}] \left( \widetilde{B}_n(X_0 f(s)) \right) ds, \text{ for } t \geq \sigma, \\
&= U(t - \sigma) \xi(\sigma) + \lim_{n \rightarrow \infty} \int_{\sigma}^t U(t - s) \left( \widetilde{B}_n(X_0 f(s)) \right) ds, \text{ for } t \geq \sigma.
\end{aligned}$$

From Theorem 10, we conclude that the function

$$\left( \Phi \zeta(t) + \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^{\mathcal{S}}(t - s) \Pi^{\mathcal{S}} \left( \widetilde{B}_n(X_0 f(s)) \right) ds \right) (0)$$

is an integral solution of Equation (1).

## 5 Almost periodic solutions

Recall that a continuous function  $v$  is called almost periodic if and only if the subset  $\{v_{\tau}, \tau \in \mathbb{R}\}$  is precompact in  $BC(\mathbb{R}, X)$ , when  $v_{\tau}$  is defined by

$$v_{\tau}(s) = v(\tau + s) \text{ for all } s \in \mathbb{R}.$$

In this section we assume that

(H<sub>4</sub>)  $f$  is almost periodic.



**Theorem 23.** Assume that  $(\mathbf{A})$ ,  $(\mathbf{B})$ ,  $(\mathbf{D}_1)$ ,  $(\mathbf{D}_2)$ ,  $(\mathbf{H}_2)$ ,  $(\mathbf{H}_3)$  and  $(\mathbf{H}_4)$  hold. If Equation (1) has a bounded solution on  $\mathbb{R}^+$ , then it has an almost periodic solution on  $\mathbb{R}$ .

**Proof.**

Since  $f$  is almost periodic, then  $t \rightarrow \langle x^*, f(t) \rangle$  is also almost periodic.

**Lemma 24.** [10, pp.86] Let  $D$  be a  $d \times d$  matrix and  $g : \mathbb{R} \rightarrow \mathbb{R}^n$  be an almost periodic function. Then the following ordinary differential equation

$$\dot{\zeta}(t) = D\zeta(t) + g(t), \quad t \in \mathbb{R},$$

has an almost periodic solution on  $\mathbb{R}$  if and only if it has a bounded solution on  $\mathbb{R}^+$ .

Consider the reduced ordinary differential Equation (6) associated to Equation (1). Let  $u$  be a bounded solution of Equation (1) on  $\mathbb{R}^+$ . Then  $\langle \Psi, u_t \rangle$  is a bounded solution of equation (6) on  $\mathbb{R}^+$ . By Lemma 24, we deduce that the ordinary differential equation (6) has an almost periodic solution  $\zeta$ , and from Theorem 22, the function  $x$  given by

$$x(t) = \left( \Phi\zeta(t) + \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^{\mathcal{S}}(t-s) \Pi^{\mathcal{S}} \left( \widetilde{B}_n(X_0 f(s)) \right) ds \right) (0), \quad t \in \mathbb{R} \quad (10)$$

is a bounded integral solution of Equation (1) on  $\mathbb{R}$ . To complete the proof, we will show that the function

$$t \rightarrow \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^{\mathcal{S}}(t-s) \Pi^{\mathcal{S}} \left( \widetilde{B}_n(X_0 f(s)) \right) ds$$

is almost periodic. Let  $(\tau_m)_{m \geq 0}$  be any real sequence. Then it possesses a subsequence  $(\tau_{m_k})_{k \geq 0}$  such that  $f(t + \tau_{m_k})$  converges uniformly on  $\mathbb{R}$  to some function  $h$ . By dominated convergence Theorem, we can that

$$\lim_{n \rightarrow +\infty} \int_{-\infty}^{t+m_k} U^{\mathcal{S}}(t+m_k-s) \Pi^{\mathcal{S}} \left( \widetilde{B}_n(X_0 f(s)) \right) ds$$

converges in  $k$  uniformly for  $t \in \mathbb{R}$  to the following function

$$t \rightarrow \lim_{n \rightarrow +\infty} \int_{-\infty}^t U^{\mathcal{S}}(t-s) \Pi^{\mathcal{S}} \left( \widetilde{B}_n(X_0 h(s)) \right) ds,$$

this property holds for every sequence  $(\tau_m)_{m \geq 0}$ . This implies that the function  $\xi$  given by (10) is almost periodic.  $\square$

## 6 Application to the Lotka-Volterra model

In order to illustrate the previous results, we consider the following Lotka-Volterra model with diffusion, which has been studied in periodic case in [5]:

$$\begin{cases} \frac{\partial}{\partial t} v(t, x) &= \frac{\partial^2}{\partial x^2} v(t, x) + a \int_{-\infty}^0 \eta(\theta) v(t + \theta, x) d\theta + F(t, x), \quad t \geq 0 \text{ and } 0 \leq x \leq \pi, \\ v(t, 0) &= v(t, \pi) = 0, \quad t \geq 0, \\ v(\theta, x) &= v_0(\theta, x), \quad -\infty < \theta \leq 0, \text{ and } 0 \leq x \leq \pi, \end{cases} \quad (11)$$

where  $a$  is a positive constant,  $\eta$  is a positive function on  $(-\infty, 0]$ ,  $v_0 : (-\infty, 0] \times [0, \pi] \rightarrow \mathbb{R}$  is a continuous function. We assume that  $F : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R}$  is almost periodic function in  $t$  uniformly with respect to  $x \in [0, \pi]$ , which means that for each  $\varepsilon > 0$  there exists  $l_\varepsilon > 0$  such that every interval of length  $l_\varepsilon$  contains a number  $\tau$  with the property that

$$\sup_{t \in \mathbb{R}, x \in [0, \pi]} |F(t + \tau, x) - F(t, x)| < \varepsilon.$$

Let  $X := C([0, \pi]; \mathbb{R})$  be the space of continuous function from  $[0, \pi]$  to  $\mathbb{R}$  endowed with the uniform norm topology. Consider the operator  $A : D(A) \subset X \rightarrow X$  defined by

$$\begin{cases} D(A) = \{z \in C^2([0, \pi]; \mathbb{R}) : z(0) = z(\pi) = 0\}, \\ Az = z''. \end{cases}$$

**Lemma 25.** [7] *A is a Hille-Yosida operator on X.*

This lemma implies that Assumption  $(\mathbf{H}_1)$  is satisfied. Moreover,

$$\overline{D(A)} = \{\psi \in C([0, \pi]; \mathbb{R}) : \psi(0) = \psi(\pi) = 0\}.$$

In [5], the authors have chosen the following phase space  $\mathcal{B} = C_\gamma$ ,  $\gamma > 0$  where

$$C_\gamma = \left\{ \phi \in C((-\infty, 0]; X) : \lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \phi(\theta) \text{ exists in } X \right\}$$

with the following norm

$$\|\phi\|_\gamma = \sup_{-\infty < \theta \leq 0} e^{\gamma\theta} |\phi(\theta)|, \text{ for } \phi \in C_\gamma.$$

**Lemma 26.** [13] *The space  $C_\gamma$ , for  $\gamma > 0$  is a uniform fading memory space satisfying  $(\mathbf{D}_1)$  and  $(\mathbf{D}_2)$ .*

We make the following assumptions as in [5].

- (E<sub>1</sub>) :  $\eta(\cdot)e^{-\gamma\cdot}$  is integrable on  $(-\infty, 0]$ .
- (E<sub>2</sub>) :  $\lim_{\theta \rightarrow -\infty} e^{\gamma\theta} v_0(\theta, x)$  exists uniformly for  $x \in [0, \pi]$ , and  $v_0(0, 0) = v_0(0, \pi) = 0$ .

We define

$$\begin{cases} (L\phi)(x) = a \int_{-\infty}^0 \eta(\theta) \phi(\theta)(x) d\theta, & x \in [0, \pi], \phi \in \mathcal{B}, \\ f(t)(x) = F(t, x), & t \geq 0, x \in [0, \pi]. \end{cases}$$

Then  $L$  be a bounded linear operator from  $\mathcal{B}$  into  $X$ , and  $f : \mathbb{R}^+ \rightarrow X$  be a continuous function. If we put

$$\begin{cases} u(t)(x) = v(t, x), & t \geq 0, x \in [0, \pi], \\ \phi(\theta)(x) = v_0(\theta, x), & \theta \leq 0, x \in [0, \pi]. \end{cases}$$

Then, Equation (11) takes the following abstract form:

$$\begin{cases} \frac{d}{dt}u = Au(t) + Lu_t + f(t), & t \geq 0, \\ u_0 = \phi. \end{cases} \quad (12)$$

Let  $A_0$  be the part of  $A$  on  $\overline{D(A)}$  given by

$$\begin{cases} D(A) = \{z \in C^2([0, \pi]; \mathbb{R}) : z(0) = z(\pi) = z''(0) = z''(\pi) = 0\}, \\ Az = z''. \end{cases}$$

$A_0$  generates a compact  $C_0$ -semigroup on  $\overline{D(A)}$  and Hypothesis (H<sub>2</sub>) is satisfied.

- (E<sub>3</sub>) : There exists  $d \in (0, 1)$  such that  $0 < a \int_{-\infty}^0 \eta(\theta) d\theta < 1 - d$ .

**Theorem 27.** [5, pp 279] If (E<sub>1</sub>), (E<sub>2</sub>) and (E<sub>3</sub>) hold, then Equation (12) has a bounded integral solution on  $\mathbb{R}^+$ . Moreover, if  $F$  is  $\omega$ -periodic in  $t$ , then Equation (12) has an  $\omega$ -periodic integral solution.

Since  $F$  is almost periodic in  $t$  uniformly for  $x \in [0, \pi]$ , then the function  $f$  is almost periodic from  $\mathbb{R}$  to  $X$ . Finally all conditions of Theorem 23 are satisfied. Consequently we establish the almost periodic version of Theorem 27.

**Theorem 28.** Assume that (E<sub>1</sub>), (E<sub>2</sub>) and (E<sub>3</sub>) hold. Then Equation (12) has an almost periodic solution.

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